Student Number

# Knox Grammar School

## 2009

**Trial Higher School Certificate Examination**

# Mathematics Extension 2

**This paper MUST NOT be removed from the examination room**

**General Instructions**

- Reading time – 5 minutes
- Working time - 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

**Subject Teachers**
Mr R. Maxwell
Mrs M. Wassef

**Number of Students in Course: 40**

**Number of Writing Booklets Per Student (Eight Page) 8**

**Total Marks – 120**

- Attempt Questions 1 – 8
- Answer each question in a separate writing booklet
- All questions are of equal value

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**Total marks – 120**
**Attempt Questions 1–8**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

**Question 1** (15 marks)

(a) Find $\displaystyle\int \frac{dx}{\sqrt{16x^2-1}}$ 2

(b) Evaluate $\displaystyle\int_1^e x\ln x\,dx$ 3

(c) (i) Find real numbers $a$ and $b$ such that 2

$$\frac{5x^2+x+8}{(x+1)(x^2+3)} \equiv \frac{a}{x+1} + \frac{bx-1}{x^2+3}$$

(ii) Hence find $\displaystyle\int \frac{5x^2+x+8}{(x+1)(x^2+3)}\,dx$ 2

(d) Find $\displaystyle\int \tan^3 x\,dx$ 2

(e) Using a suitable substitution, or otherwise, evaluate: 4

$$\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$$

**End of Question 1**



**Question 2** (15 marks) Use a SEPARATE writing booklet. Marks

(a) Let $\alpha = 1-\sqrt{3}\,i$.

(i) Find the exact value of $|\alpha|$ and $\arg\alpha$. 2

(ii) Hence express $\left(1-\sqrt{3}\,i\right)^{10}$ in modulus-argument form. 1

(b) Express $\sqrt{7-24i}$ in the form $a+ib$, where $a$ and $b$ are real. 3

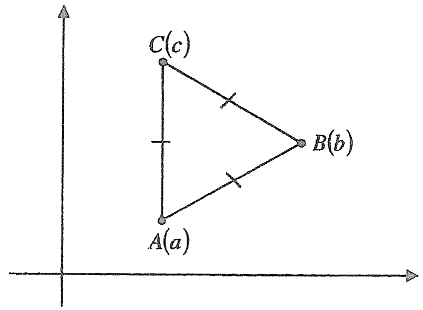
(c) Sketch the region in the complex plane where the two inequalities $0 \le \operatorname{Arg}(z) \le \frac{3\pi}{4}$ and $|z-2i| \ge |z|$ both hold. 3

(d) Sketch the locus of $z$ satisfying $|z-3|+|z+3| = 10$. Show any intercepts with the axes. 3

**Question 2 continues on page 4**

Question 2 (continued)

(e)




The points $A$, $B$ and $C$ on the Argand diagram represent the complex numbers $a$, $b$ and $c$ respectively. $\Delta ABC$ is equilateral.

Let $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Show that $\dfrac{a-b}{c-b} = w$. 1

(ii) By writing another similar expression for $w$, prove that 2

$$a^2 + b^2 + c^2 = ab + bc + ca$$

**End of Question 2**





Marks

**Question 3** (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 + 3x^2 - 5x - 2 = 0$ has roots $\alpha$, $\beta$ and $\gamma$. 2

Find a cubic equation with integer coefficients whose roots are $\dfrac{2}{\alpha}, \dfrac{2}{\beta}$ and $\dfrac{2}{\gamma}$.

(b) Consider the curve $x^2 + y^2 + xy = 3$.

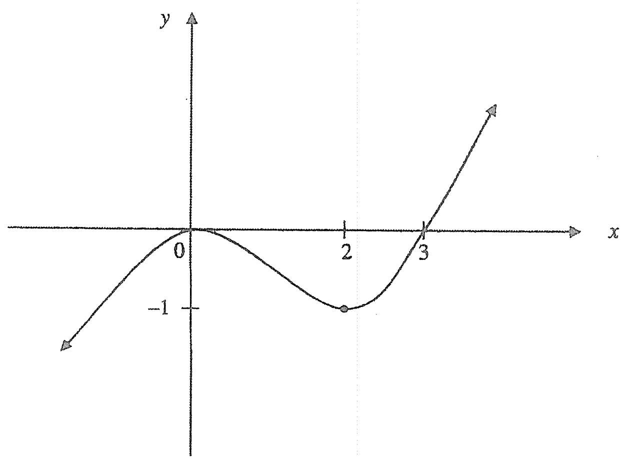
(i) Show that $\dfrac{dy}{dx} = -\left(\dfrac{2x+y}{x+2y}\right)$. 1

(ii) Hence find the coordinates of any stationary points. 2

**Question 3 continues on page 6**

Question 3 (continued) **Marks**

(c) The diagram shows the graph of $y = f(x)$ where $f(x) = \frac{1}{4}x^2(x-3)$.




On the answer page provided, draw separate sketches of the graphs of the following:

(i) $y = \frac{1}{4}x^2 \, |x-3|$ **1**

(ii) $y = \frac{1}{f(x)}$ **1**

(iii) $y^2 = -f(x)$ **2**

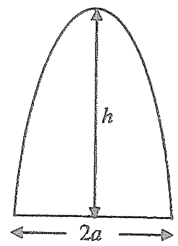
(iv) $y = \tan^{-1}(f(x))$ **2**

**Question 3 continues on page 7**



Question 3 (continued) **Marks**

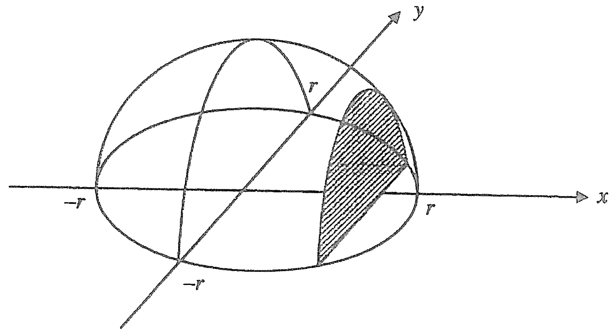
(d) (i) **1**




A parabolic segment has height $h$ and width $2a$.

Use Simpson's Rule with three function values, to show that the exact area of this segment is $\frac{4ah}{3}$.

(ii) **3**




The base of a solid is the region in the $xy$ plane enclosed by the circle $x^2 + y^2 = r^2$.

Each cross-section perpendicular to the $x$-axis is a parabolic segment with height one half its width.

Show that the volume of the solid is $\frac{16r^3}{9}$ units$^3$.

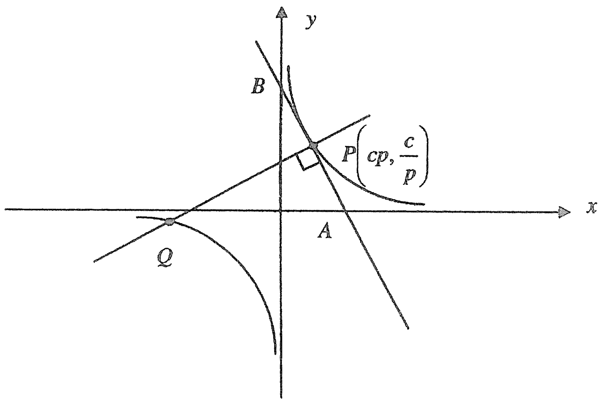
**End of Question 3**

Marks

**Question 4** (15 marks) Use a SEPARATE writing booklet.

(a) The point $P\left(cp, \frac{c}{p}\right)$ is a point on the hyperbola $xy = c^2$.

The tangent to the hyperbola at $P$ intersects the $x$ and $y$ axes at $A$ and $B$ respectively and the normal to the hyperbola at $P$ intersects the second branch at $Q$.




(i) Show that the equation of the normal at $P$ is $py - c = p^3\left(x - cp\right)$. 2

(ii) Show that the $x$ coordinates of $P$ and $Q$ satisfy the equation 2

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0$$

and hence find the coordinates of $Q$.

(iii) Given the distance $AB = 2c\sqrt{p^2 + \frac{1}{p^2}}$, show that the 2

area of $\Delta ABQ = c^2\left(p^2 + \frac{1}{p^2}\right)^2$.

(iv) Find the minimum area of $\Delta ABQ$. 1

(You may use the inequality $\frac{a}{b} + \frac{b}{a} \geq 2$ for $a, b > 0$.)

**Question 4 continues on page 9**





Question 4 (Continued) Marks

(b) An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has foci $S$ and $S'$ where $b^2 = a^2(1 - e^2)$ where $0 < e < 1$ is its eccentricity.

(i) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is 2

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

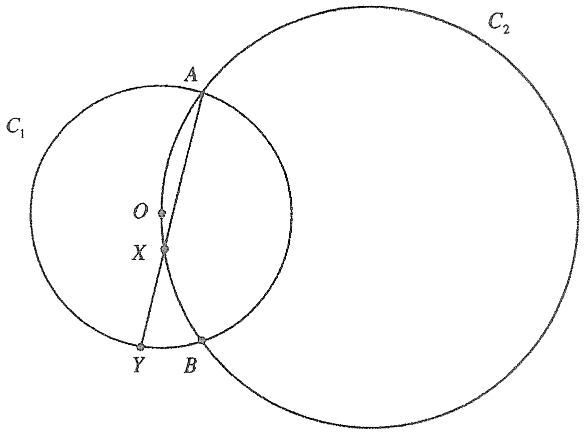
(ii) Show that $N$, the $x$-intercept of the normal has coordinates $N\left(e^2x_1, 0\right)$. 1

(iii) Use the focus/directrix definition of a conic to prove that $\frac{SN}{S'N} = \frac{SP}{S'P}$ 2

**Question 4 continues on page 10**

Question 4 (continued) **Marks**

(c)




Two circles $C_1$ and $C_2$ intersect at $A$ and $B$. $C_2$ passes through $O$, the centre of $C_1$. $X$ lies on the arc $AOB$ and $AX$ intersects $C_1$ again at $Y$.

(i) State why $\angle AOB = 2 \times \angle AYB$. **1**

(ii) Prove that $XY = XB$. **3**

**Question 5** (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that if $\alpha$ is a double root of $f(x)=0$ then $f(\alpha)=f'(\alpha)=0$. **2**

(ii) Find all roots of the equation $2x^3-5x^2-4x+12=0$ given that two of the roots are equal. **3**

(b) (i) By drawing a diagram, or otherwise, find the solutions of $z^5=1$. **2**

(ii) Show that $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$. **2**

(iii) Hence find the exact value of $\cos\frac{2\pi}{5}$. **2**

(c) 11 persons gather to play basketball by forming 2 teams of 5 to play each other. The remaining person acts as a referee.

(i) In how many ways can the teams be formed? **2**

(ii) If two particular persons are not to be in the same team, how many ways are there then to choose the teams? **2**

Marks

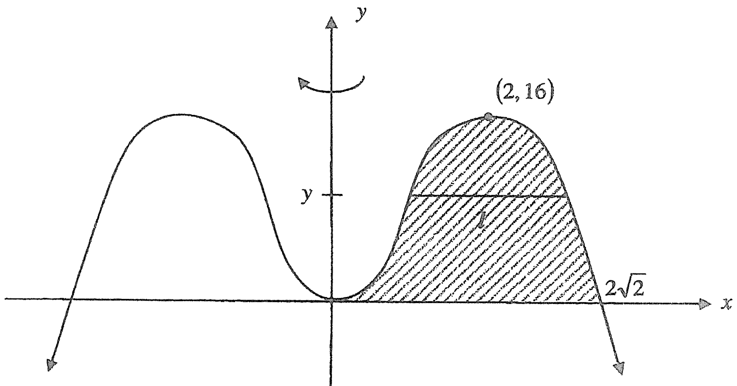
**Question 6** (15 marks) Use a SEPARATE writing booklet.

(a) The sequence $\{a_n\}$ is given by: 3

$$a_1 = 2,\ a_2 = \frac{3}{2} \text{ and } (n+1)a_{n+1} = a_{n-1} - (n-2)a_n \text{ for } n > 1.$$

Prove by induction that for $n \geq 1$, $a_n = \dfrac{n+1}{n!}$

(b) The region bound by the curve $y = 8x^2 - x^4$ and the $x$ axis in the first quadrant is rotated about the $y$ axis to form a solid. When the region is rotated, the horizontal line segment $l$ at height $y$ sweeps out an annulus.




(i) Show that the area of the annulus at height $y$ is given by $2\pi\sqrt{16-y}$. 3

(ii) Find the volume of the solid. 2

**Question 6 continues on page 13**

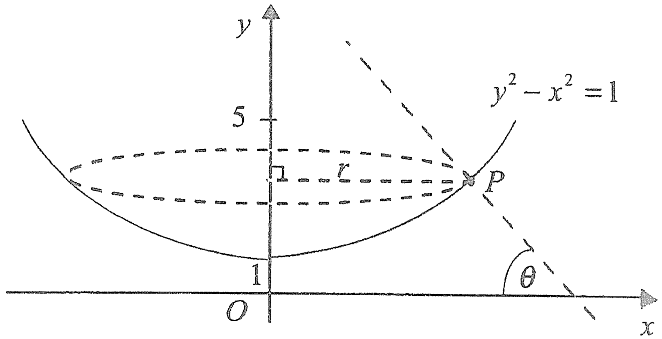




Question 6 (continued)

Marks

(c)




A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \leq y \leq 5$ through $180^\circ$ about the $y$-axis. Sometime later, a particle P of mass $m$ moves around the inner surface of the bowl in a horizontal circle with constant angular velocity $\omega$.

(i) Show that if the radius of the circle in which P moves is $r$, then the normal to the surface at P makes an angle $\theta$ with the horizontal where $\tan\theta = \dfrac{\sqrt{1+r^2}}{r}$. 3

(ii) Find expressions for the radius $r$ of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of $m, g$ and $\omega$. 2

(iii) Find the values for $\omega$ for which the described motion of P is possible. 3

**End of Question 6**

**Question 7** (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle of mass $m$ is projected vertically upwards with and initial velocity of $V$ in a medium where the resistance force $R$ to the motion has a magnitude $R=mkv$ where $v$ is the velocity of the particle after the initial projection.

(i) Show that the maximum height $h$ of the particle is given by **3**

$$h=\frac{g}{k^2}\left\{\frac{k}{g}V-\ln\left(1+\frac{k}{g}V\right)\right\}$$

(ii) Find an expression for the time $T$ of particle to reach its maximum height in terms of $V$, $k$ and $g$. **2**

(iii) After reaching its maximum height the particle returns vertically downwards towards its projection point in the same medium. Show that the downward velocity is given by $v=\frac{g}{k}\left(1-e^{-kt}\right)$ where $t$ is the time of the downward motion and give the terminal velocity of the particle. **3**

(iv) The speed of upward projection is double the terminal velocity and the particle's downward displacement $y$ from its maximum height is given by the equation $-\frac{k^2}{g}y=\frac{k}{g}v+\ln\left(1-\frac{k}{g}v\right)$. (Do NOT prove this result).

Show that the velocity of the particle on return to its projection point is given by $\frac{kv}{g}+2-\ln 3+\ln\left(1-\frac{kv}{g}\right)=0$. If a root to this equation for $\frac{kv}{g}$ lies near 0.81, use one application of "Newton's Method" to find a better approximation correct to two decimal places and deduce the percentage of the terminal velocity that the particle has acquired on return to its projection point. **3**

(v) Hence find the ratio of the time taken to reach maximum height to the time to fall from maximum height to the point of projection. **1**

(b) When an unbiased coin is tossed $2n$ times, the probability of observing $k$ heads and $2n-k$ tails is given by $P_k={}^{2n}C_k\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{2n-k}$.

(i) Show that the most likely outcome is $k=n$ **2**

(ii) Show that $P_n=\dfrac{(2n)!}{2^{2n}(n!)^2}$ **1**

**End of Question 7**

**Question 8** (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Given $I_n=\int_1^e(1-\ln x)^n\,dx$ for $n=0, 1, 2, \ldots$

(i) Show that $I_n=-1+nI_{n-1}$ for $n=1, 2, 3\ldots$ **2**

(ii) Hence evaluate $\int_1^e(1-\ln x)^3\,dx$ **2**

(iii) Show that $\frac{I_n}{n!}=e-\sum_{r=0}^{n}\frac{1}{r!}$ for $n=1, 2, 3\ldots$ **2**

(iv) Show that $0\le I_n\le e-1$ **1**

(v) Deduce that $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}=e$ **1**

(b) $\phi(x)$ and $\psi(x)$ are continuous and bounded functions.

(i) By considering $\int_0^a\{\lambda\phi(x)+\psi(x)\}^2dx$ for $a>0$ as a quadratic function in $\lambda$, show that $\left\{\int_0^a\phi(x)\psi(x)dx\right\}^2\le\int_0^a\{\phi(x)\}^2dx\times\int_0^a\{\psi(x)\}^2dx$. **4**

(ii) Hence show that $\left\{\int_0^1\phi(x)dx\right\}^2\le\int_0^1\{\phi(x)\}^2dx$. **2**

(iii) Deduce that $\left\{\int_0^1\phi(x)dx\right\}^4\le\int_0^1\{\phi(x)\}^4dx$. **1**

**End of Paper**

1. WAM  5. WAM
2. MAX  6. MAX
3. HAN  7. WAM
4. DIN  8. MAX

# 2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS

| Suggested Solution (s) | Comments |
|---|---|

## Question 1

a) $\int \frac{dx}{\sqrt{16x^2-1}} = \frac{1}{4}\int \frac{dx}{\sqrt{x^2-(\frac{1}{4})^2}}$ — ✓ coeff of $\frac{1}{4}$

$= \frac{1}{4}\ln\left(x+\sqrt{x^2-\frac{1}{16}}\right)$

$= \frac{1}{4}\ln\left(4x+\sqrt{16x^2-1}\right)+C$ — ✓ answer or equiv.

b) $\int_1^e x\ln x$  $u = \ln x$, $v' = x$, $u' = \frac{1}{x}$, $v = \frac{x^2}{2}$

$= \left[\frac{x^2\ln x}{2}\right]_1^e - \int_1^e \frac{x}{2}\,dx$ ✓

$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$ ✓

$= \frac{e^2}{4}+\frac{1}{4}$ ✓

c) i) $a=3$ ✓ $b=2$ ✓

ii) $\int \frac{5x^2+x+8}{(x+1)(x^2+3)}\,dx = \int \frac{3}{x+1}+\frac{2x-1}{x^2+3}\,dx$

$= 3\ln|x+1| + \ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$ ✓ ✓

d) $\int \tan^3 dx = \int \tan x(\sec^2 x - 1)\,dx$

$= \int \sec^2 x\tan x - \frac{\sin x}{\cos x}\,dx$

$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$ ✓ ✓

e) $\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$  let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$; $x=0,\ \theta=0$; $x=2,\ \theta=\frac{\pi}{2}$ ✓

$= \int_0^{\frac{\pi}{2}} \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}} 4\sin^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} 2(1-\cos 2\theta)\, d\theta$ ✓

$= \left[2\theta - \sin 2\theta\right]_0^{\frac{\pi}{2}} = \pi$ ✓

## Question 2

a) i) $|\alpha| = 2$  $\arg(\alpha) = -\frac{\pi}{3}$

ii) $(1-\sqrt{3}i)^{10} = \left(2\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^{10}$

$= 2^{10}\operatorname{cis}\left(-\frac{10\pi}{3}\right)$

$= 2^{10}\operatorname{cis}\left(\frac{2\pi}{3}\right)$ ✓

b) let $(a+bi)^2 = 7-24i$

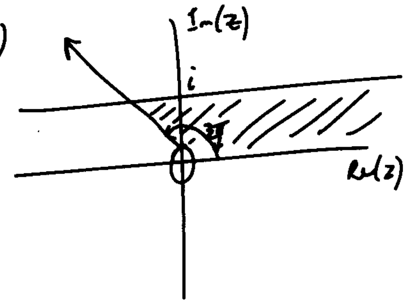
$\therefore a^2-b^2+2abi = 7-24i$ ✓

ie, $a^2-b^2=7$  $2ab=-24$, $ab=-12$

by inspection $a=\pm 4$, $b=\mp 3$  $a,b\in\mathbb{R}$

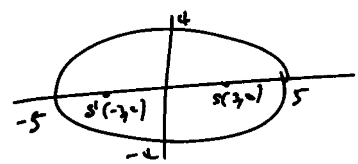
$\therefore \sqrt{7-24i} = 4-3i,\ -4+3i$ ✓ ✓

c)




✓ $0 \le \arg(z) \le \frac{3\pi}{4}$
✓ $|z-2i| = |z|$
✓ region
deduct 1 if origin included.

d)




✓ ellipse with foci $(\pm 3, 0)$
✓ x-intercepts $\pm 5$
✓ y-intercepts $\pm 4$

e) i) $\vec{BA} = (a-b)$, $\vec{BC} = c-b$

Since $\vec{BA} = \vec{BC}\times\operatorname{cis}\left(\frac{\pi}{3}\right)$ ✓

$(a-b) = (c-b)\times\omega$

$\therefore \omega = \frac{a-b}{c-b}$

ii) Similarly $\frac{c-a}{b-a} = \omega$ ✓

$\therefore \frac{a-b}{c-b} = \frac{c-a}{b-a} \Rightarrow (a-b)(b-a) = (c-a)(c-b)$

$ab-a^2-b^2+ab = c^2-cb-ac+ab$

$\therefore a^2+b^2+c^2 = ab+bc+ca$

✓ progress to.

2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|

## Question 3

a) let $y = \frac{2}{x}$

$\left(\frac{2}{x}\right)^3 + 3\left(\frac{2}{x}\right)^2 - 5\left(\frac{2}{x}\right) - 2 = 0$ ✓

$\Rightarrow y^3 + 5y^2 - 6y + 4 = 0$ ✓

b) $x^2 + y^2 + xy = 3$

i) $2x + 2y \cdot \frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right) = 0$ ✓

$(x + 2y)\frac{dy}{dx} = -2x - y$

$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$

ii) For stat. pts $\frac{dy}{dx} = 0$

$y = -2x$

$\therefore x^2 + (-2x)^2 + x(-2x) = 3$

$3x^2 = 3$

$x = \pm 1$

$y = \mp 2$

$\therefore$ stat pts $(1,-2)$ ✓ and $(-1,2)$ ✓

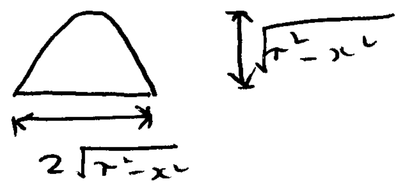
c) see separate pages

d) i) $A = \frac{h}{3}\left[y_1 + 4y_2 + y_3\right)$

$= \frac{a}{3}\left[0 + 4h + 0\right)$

$= \frac{4ah}{3}$ ✓

ii) Cross-section:




$\therefore$ Area $= \frac{4ah}{3} = \frac{4}{3}(r^2 - x^2)$

$V = 2\int_0^r \frac{4}{3}(r^2 - x^2)\,dx$ ✓

$= \frac{8}{3}\left[r^2x - \frac{x^3}{3}\right]_0^r$ ✓

$= \frac{16r^3}{9}\ u^3$ ✓

**HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

# Mathematics Extension 2

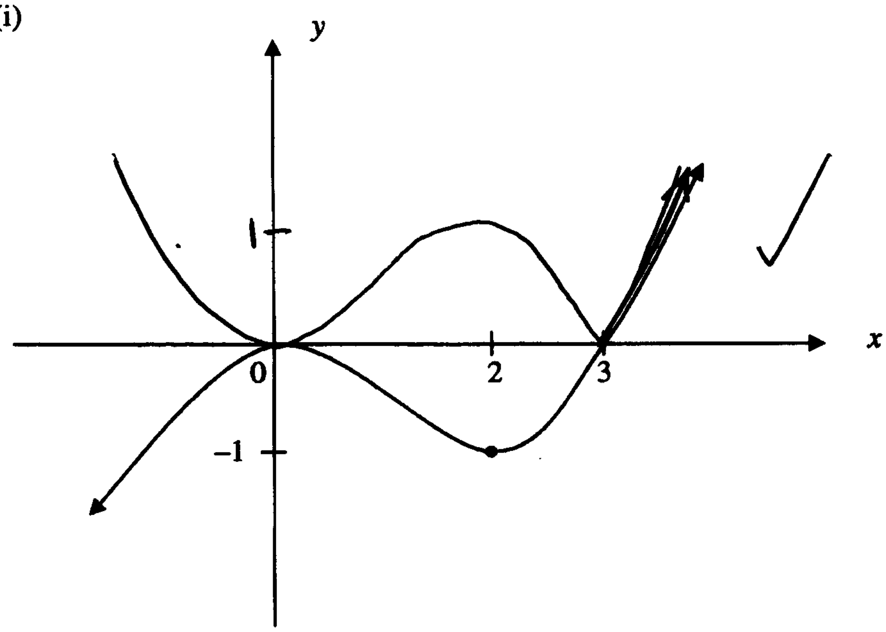
**Questions 3 (c)**

Centre Number

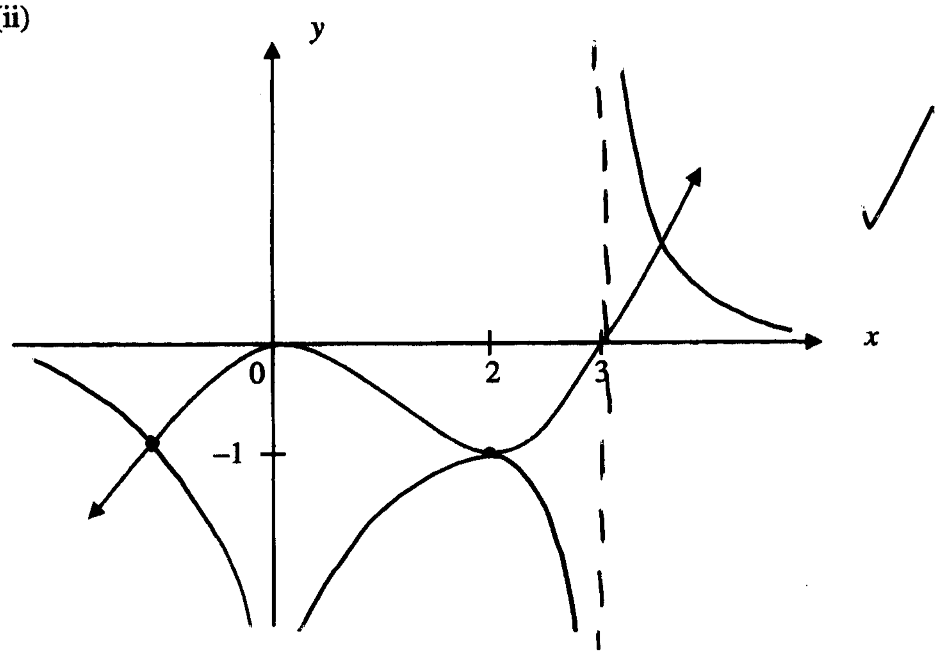
| S | O | L | U | T | I | O | N | S |
|---|---|---|---|---|---|---|---|---|

Student Number

(i)



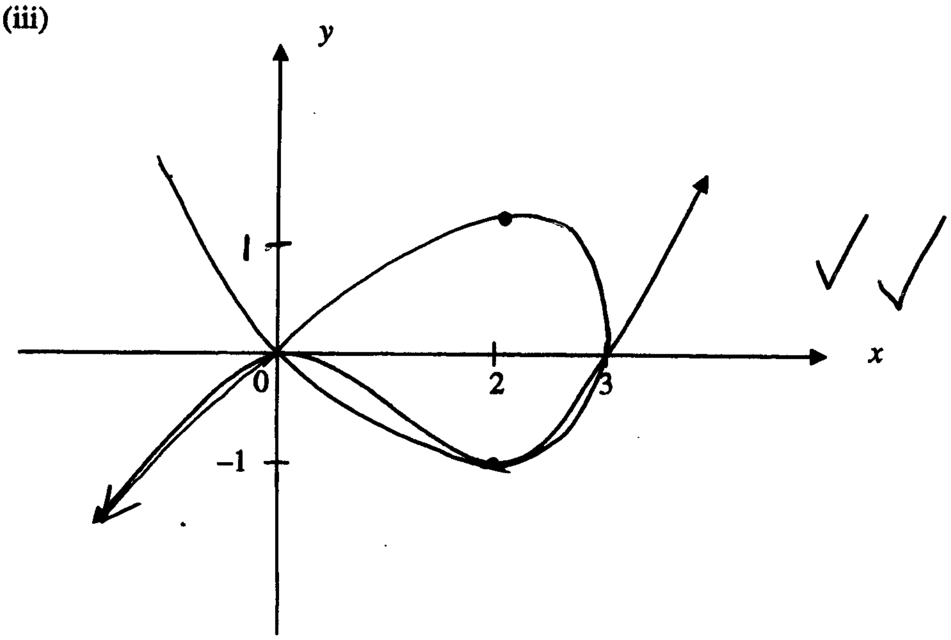

(ii)



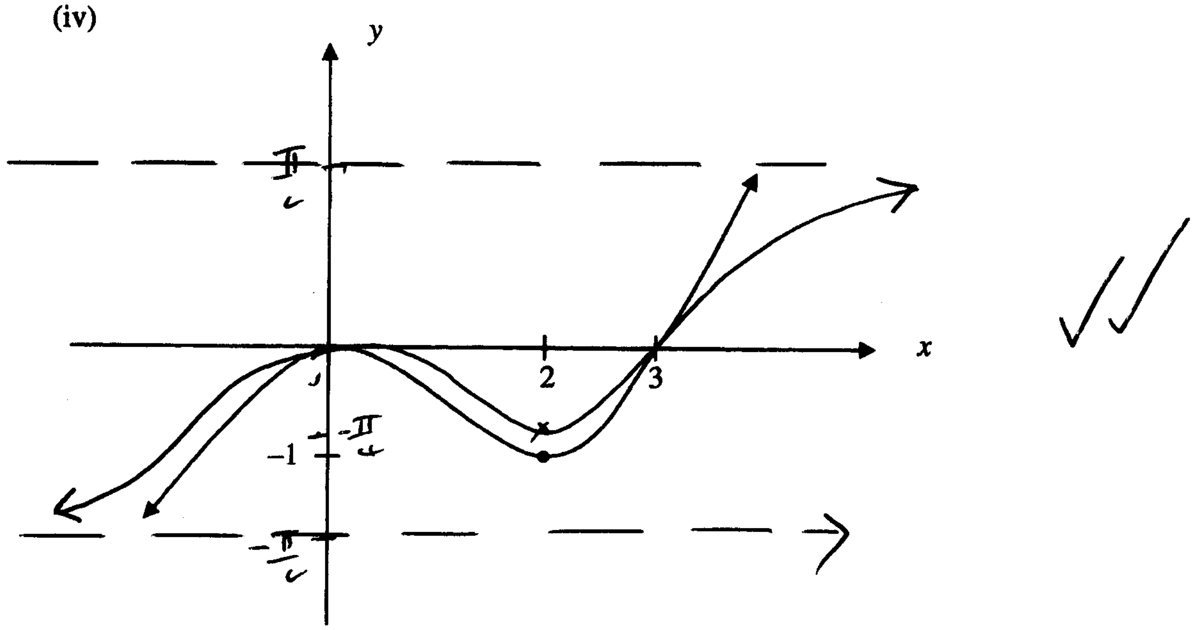

**Question 3(c) continues on next page**

Question 3 (continued)

(c) (iii)




(iv)



## Question 4

a) i)

$$xy = c^2$$

$$x \cdot \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

at $P(cp, \frac{c}{p})$ $\quad \frac{dy}{dx} = \frac{-\frac{c}{p}}{cp}$

$$\therefore m_T = -\frac{1}{p^2}$$

$$m_T = p^2 \quad ✓$$

$$\therefore y - \frac{c}{p} = p^2(x - cp)$$

$$py - c = p^3(x - cp) \quad ✓$$

ii) Solving $\quad y = \frac{c^2}{x}$ — ①

$$py - c = p^3(x - cp) \quad — ②$$

$$p \cdot \frac{c^2}{x} - c = p^3(x - cp)$$

$$pc^2 - cx = p^3x^2 - cp^4x$$

✓ - correct progress to equation

$\div p^3$: $\quad \frac{c^2}{p^2} - \frac{c}{p^3}x = x^2 - cpx$

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0$$

Let $x$ coord of $Q$ be $\alpha$

$$\therefore \alpha + cp = c\left(p - \frac{1}{p^3}\right) \quad \text{sum of roots}$$

$$\therefore \alpha = -\frac{c}{p^3} \quad \therefore Q\left(-\frac{c}{p^3}, -cp^3\right) ✓$$

iii) $AB = 2c\sqrt{p^2 + \frac{1}{p^2}}$ (given)

$$\therefore PQ = \sqrt{\left(cp + \frac{c}{p^3}\right)^2 + \left(\frac{c}{p} + cp^3\right)}$$

$$= c\sqrt{p^2 + \frac{2}{p^2} + \frac{1}{p^6} + \frac{1}{p^2} + 2p^2 + p^6}$$

$$= c\sqrt{p^6 + 3p^2 + \frac{3}{p^2} + \frac{1}{p^6}}$$

$$= c\sqrt{\left(p^2 + \frac{1}{p^2}\right)^3}$$

$$\text{Area } \Delta ABQ = \frac{1}{2} \times AB \times PQ$$

$$= \frac{1}{2} \times 2c\sqrt{p^2 + \frac{1}{p^2}} \times c\sqrt{\left(p^2 + \frac{1}{p^2}\right)^3}$$

$$= c^2\left(p^2 + \frac{1}{p^2}\right)^2$$

iv) if $\frac{a}{b} + \frac{b}{a} \geq 2$

$$p^2 + \frac{1}{p^2} \geq 2$$

$$\therefore \text{Area } \Delta ABQ \geq c^2 \times 2^2$$

$$\geq 4c^2$$

$\therefore$ Min. Area is $4c^2$.

(b) i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

at $P(x_1, y_1)$ $\frac{dy}{dx} = -\frac{b^2x_1}{a^2y_1}$ ✓

$\therefore$ gradient of normal $= +\frac{a^2y_1}{b^2x_1}$ ✓

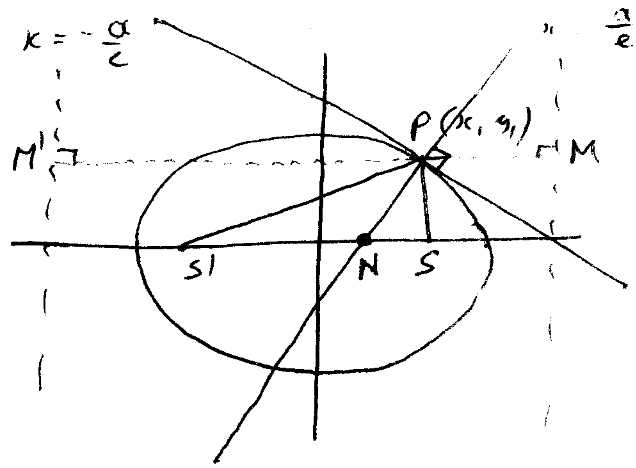
$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$$

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

ii) at $y = 0$

$$a^2x = x_1(a^2 - b^2)$$
$$a^2x = x_1(a^2 - a^2(1-e^2))$$
$$a^2x = x_1a^2e^2$$
$$\therefore x = e^2x_1$$ ✓

$\therefore N(e^2x_1, 0)$




iii) $R.H.S = \frac{PS}{PS'}$

$$= \frac{e\,PM}{e\,PM'}$$

$$= \frac{\frac{a}{e} - x_1}{\frac{a}{e} + x_1}$$ ✓

$$= \frac{a - ex_1}{a + ex_1}$$ ✓

$L.H.S = \frac{SN}{S'N}$

$$= \frac{ae - e^2x_1}{ae - e^2x_1} = \frac{\cancel{e}(a - ex_1)}{\cancel{e}(a - ex_1)}$$

✓ $\therefore \frac{SN}{S'N} = \frac{SP}{S'P}$

4(c) i) If $\angle XYB = \alpha$

$\angle AOB = 2\alpha$ ($\angle$ at centre is double $\angle$ at circumference standing on same arc).

$\angle AXB = \angle AOB = 2\alpha$ ($\angle$'s in same segment are equal).

$\therefore \angle XBY = \angle AXB - \angle XYB$

$= 2\alpha - \alpha$

$= \alpha$ (Ext $\angle$. of $\Delta$ equals sum of 2 opp. int. $\angle$'s)

$\therefore XY = XB$ (sides opposite equal $\angle$'s in a $\Delta$ are equal)

## Question 5

(a) (i) If $\alpha$ is a double root of $f(x)=0$, then $f(x)$ can be written:

$$f(x) = (x-\alpha)^2 . Q(x)$$
$$f'(x) = 2(x-\alpha).Q(x) + (x-\alpha)^2 . Q'(x)$$
$$= (x-\alpha)\left[2Q(x) + (x-\alpha).Q'(x)\right]$$
$$= (x-\alpha). \Phi(x)$$

$f(\alpha) = (\alpha-\alpha)^2 . Q(\alpha)$ $\quad$ $f'(\alpha) = (\alpha-\alpha).\Phi(\alpha)$

$= 0 . Q(\alpha)$ $\quad$ $= 0 . \Phi(\alpha)$

$= 0$

$\therefore f(\alpha) = f'(\alpha) = 0$

(ii) let $f(x) = 2x^3 - 5x^2 - 4x + 12$

so $f'(x) = 6x^2 - 10x - 4 = 0$

$3x^2 - 5x - 2 = 0$

$(3x+1)(x-2) = 0$

$x = -\frac{1}{3}, 2$

and $f(2) = 0$ $\therefore x = 2$ is the double root

so $2 + 2 + \beta = \frac{5}{2}$

$\beta = -\frac{3}{2}$

$\therefore$ Roots are $2, 2, -\frac{3}{2}$

5b) i) $z_0 = 1$

$z_1 = \text{cis}\,\frac{2\pi}{5}$

$z_2 = \text{cis}\,\frac{4\pi}{5}$

$z_3 = \text{cis}\,-\frac{4\pi}{5}$

$z_4 = \text{cis}\,-\frac{2\pi}{5}$

All roots

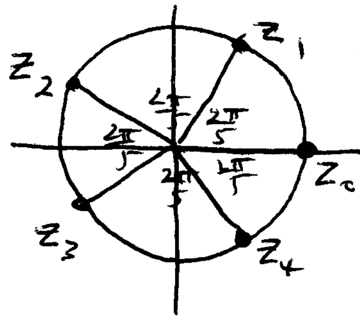



diagram or otherwise

ii) Sum of roots $= -\frac{b}{a}$

$\therefore z_0 + z_1 + z_2 + z_3 + z_4 = -\frac{0}{1}$ $\quad z^5 - 1 = 0$

Noting $\bar{z}_1 = z_4 \quad \bar{z}_2 = z_3$

$1 + z_1 + \bar{z}_1 + z_2 + \bar{z}_2 = 0$

Note $z + \bar{z} = 2\,\text{Re}(z)$

$\therefore 1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

iii) $\cos\frac{2\pi}{5} + 2\cos^2\frac{2\pi}{5} - 1 = -\frac{1}{2}$

$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} + 1 = 0$

$\therefore \cos\frac{2\pi}{5} = \frac{-1 \pm \sqrt{5}}{4}$ $\quad$ but $\cos\frac{2\pi}{5} > 0$

$\therefore \cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$

c i) $\frac{{}^{11}C_5 \times {}^{6}C_5 \times {}^{1}C_1}{2} = 1386$

ii) ${}^{9}C_4 \times {}^{5}C_4 \times {}^{1}C_1 + \frac{{}^{2}C_1 \times {}^{10}C_5 \times {}^{5}C_5}{2}$

$= 630 + 252 = 882.$

**2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS**

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|
| Question 6 a)<br>When $n=1$, $\frac{1+1}{1!} = 2 = a_1$<br>when $n=2$ $\frac{2+1}{2!} = \frac{3}{2} = a_2$<br>$\therefore$ Statement True for $n=1,2$<br>Assume the statement true for $n=k-1$ and $n=k$<br>ie $a_{k-1} = \frac{k}{(k-1)!}$<br>$a_k = \frac{k+1}{k!}$<br>When $n=k+1$<br>$L.H.S = a_n$<br>$= a_{k+1}$<br>$= \frac{a_{k-1} - (k-2)a_k}{(k+1)}$<br>$= \frac{\frac{k}{(k-1)!} - (k-2)\frac{k+1}{k!}}{(k+1)}$<br>$= \frac{k^2 - (k^2-k-2)}{k!(k+1)}$<br>$= \frac{k+2}{(k+1)!}$<br>$= \frac{n+1}{n!}$ when $n=k+1$ | ✓<br>✓<br>as | ✓<br>$(n+1)a_{n+1} = a_{n-1} - (n-2)a_n$<br>If the statement true for $n=k-1, k$ then proved true for $n=k+1$.<br>Since true for $n=1,2$,<br>$\therefore$ true for $n=3,4,5,\ldots$<br>$\therefore$ true for all $n \geq 1$. | |

1. WAM
2. MAX
3. HAR
4. DIN
5. WAM
6. WAM
7. MAX
8. MAX

**2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS**

| Suggested Solution (s) | Comments |
|---|---|

6 b) i) $y = 8x^2 - x^4$

$x^4 - 8x^2 = -y$

$(x^2-4)^2 = 16 - y$

$x^2 = \pm\sqrt{\pm\sqrt{16-y}+4}$

let $x_1$ and $x_2$ be endpoints of $l$ with $0 \le x_1 \le x_2$

$\therefore x_2 = +\sqrt{+\sqrt{16-y}+4}$

$x_1 = -\sqrt{-\sqrt{16-y}+4}$

$\therefore \text{Area} = \pi(x_2^2 - x_1^2)$

$= \pi\left(\sqrt{16-y}+4 - (-\sqrt{16-y}+4)\right)$

$= 2\pi\sqrt{16-y}$

ii) $V = \int_0^{16} 2\pi\sqrt{16-y}\,dy$

$= \left[-\frac{4\pi}{3}(16-y)^{\frac{3}{2}}\right]_0^{16}$

$= \frac{4^4\pi}{3}\ u^3$

c) i) $y^2 - x^2 = 1$

$\therefore 2y\frac{dy}{dx} - 2x = 0$

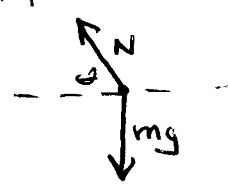
$\therefore \frac{dy}{dx} = \frac{x}{y}$

at $P(r, \sqrt{1+r^2})$

$\therefore m_{\text{NORMAL}} = -\frac{\sqrt{1+r^2}}{r}$

$\therefore \tan\theta = \frac{\sqrt{1+r^2}}{r}$

ii) Forces on P




Resolving Vertically and Horizontally

$N\sin\theta = mg$ — ①  $N\cos\theta = mr\omega^2$ — ②

① ÷ ②  $\tan\theta = \frac{g}{r\omega^2}$

$\therefore \frac{\sqrt{1+r^2}}{r} = \frac{g}{r\omega^2}$

$\therefore r = \frac{\sqrt{g^2-\omega^4}}{\omega^2}$

From ②: $N^2 = m^2r^2\omega^4\sec^2\theta$

$= m^2\omega^4(r^2 + r^2\tan^2\theta)$

$= m^2\omega^4\left(\frac{g^2}{\omega^4} - 1 + \frac{g^2}{\omega^4}\right)$

$= m^2(2g^2 - \omega^4)$

$\therefore N = \sqrt{2g^2 - \omega^4}$

iii) $\omega^4 \le g^2 \quad \therefore \omega \le \sqrt{g}$

$y \le 5 \quad \sqrt{1-r^2} \le 5 \quad \frac{g}{\omega^2} \le 5$

$\therefore \sqrt{\frac{g}{5}} \le \omega \le \sqrt{g}$

2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|
| Question 7<br>a) i) For upward motion<br>$x\uparrow \quad mg\downarrow \quad R\downarrow$<br>$F = m\ddot{x} = -mg - R$<br>$\therefore \ddot{x} = -(g+kv)$<br>$v\frac{dv}{dx} = -(g+kv)$<br>$\frac{dx}{dv} = \frac{-v}{g+kv}$<br>$k\frac{dx}{dv} = -1 + \frac{g}{g+kv}$<br>$\therefore \frac{k^2}{g}\frac{dx}{dv} = -\frac{k}{g} + \frac{k}{g+kv}$<br>$\therefore \frac{k^2}{g}x = -\frac{k}{g}v + \ln(g+kv) + C$<br>at $x=0$, $v=V$<br>$\therefore C = \frac{k}{g}V - \ln(g+kV)$<br>$\therefore \frac{k^2}{g}x = \frac{k}{g}(V-v) - \ln\left(\frac{g+kV}{g+kv}\right)$<br>or equivalent<br>at $x=h$, $v=0$<br>$\therefore h = \frac{g}{k^2}\left(\frac{k}{g}V - \ln\left(1+\frac{k}{g}V\right)\right)$ ✓ | ✓ | ii) $\ddot{x} = -(g+kv)$<br>$\frac{dv}{dt} = -(g+kv)$<br>$\therefore \frac{dt}{dv} = -\frac{1}{g+kv}$<br>$t = -\frac{1}{k}\ln(g+kv) + C$<br>at $t=0$, $v=V$<br>$\therefore t = \frac{1}{k}\ln\left(\frac{g+kV}{g+kv}\right)$<br>or equivalent<br>at $v=0$, $t=T$<br>$\therefore$<br>$T = \frac{1}{k}\ln\left(1+\frac{k}{g}V\right)$<br>or equivalent ✓<br>iii) Downward motion.<br>$\downarrow mg \quad \uparrow R$<br>$\therefore F = m\ddot{x} = mg - mkv$<br>$\frac{dv}{dt} = g - kv$<br>$\frac{dt}{dv} = \frac{1}{g-kv}$<br>$t = -\frac{1}{k}\ln(g-kv) + c$<br>at $t=0$, $v=0$ $\therefore$ | ✓<br><br><br><br><br><br>$C = \frac{1}{k}\ln g$ ✓ |

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|
| part iii) continued<br>$\therefore t = \frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$<br>or $-kt = \ln\left(\frac{g-kv}{g}\right)$<br>$\frac{g-kv}{g} = e^{-kt}$<br>$1 - \frac{k}{g}v = e^{-kt}$ ✓<br>$\therefore v = \frac{g}{k}(1-e^{-kt})$<br>at $t \to \infty$, $e^{-kt} \to 0$<br>$v \to \frac{g}{k}$ (Terminal v) ✓<br>iv) If $V = \frac{2g}{k}$<br>$h = \frac{g}{k^2}(2 - \ln 3)$ from part i)<br>and sub in $-\frac{k^2}{g}y = \frac{kv}{g} + \ln(1-\frac{k}{g}v)$<br>$\therefore \frac{k^2}{g}\cdot\frac{g}{k^2}(2-\ln 3) = \frac{kv}{g} + \ln(1-\frac{k}{g}v)$<br>$\therefore \frac{kv}{g} + 2 - \ln 3 + \ln(1-\frac{kv}{g}) = 0$ ✓ | | let $u = \frac{kv}{g}$<br>$\therefore f(u) = u + 2 - \ln 3 + \ln(1-u)$<br>$f'(u) = 1 - \frac{1}{1-u}$<br>if $u_1 = 0.81$<br>$u_2 = 0.81 - \frac{0.81+2-\ln 3+\ln(0.19)}{1-\frac{1}{0.19}}$<br>$\doteq 0.82$ (2 d.p.) ✓<br>$\therefore \frac{kv}{g} \doteq 0.82$<br>and $v \doteq 0.82\frac{g}{k}$<br>or 82% of terminal velocity on return ✓<br>v) From part iii)<br>$kt = -\ln(1-\frac{kv}{g})$<br>at $\frac{kv}{g} = 0.82$ $kT = -\ln(1-0.82)$<br>$= -\ln(.18)$<br>From part ii)<br>$V = \frac{2g}{k} \Rightarrow kT = \ln 3$<br>$\therefore \frac{-\ln(.18)}{\ln 3} \doteq 0.64$ ✓ | |

**2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS**

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|

7 (b) i)

$$\frac{P_{k+1}}{P_k} = \frac{\dfrac{\cancel{(2n)!}}{(k+1)!(2n-k-1)!}\left(\frac{1}{2}\right)^{k+1}\left(\frac{1}{2}\right)^{2n-k-1}}{\dfrac{\cancel{(2n)!}}{k!(2n-k)!}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{2n-k}}$$

$$= \frac{k!(2n-k)!}{(k+1)!(2n-k-1)!} \cdot \frac{\left(\frac{1}{2}\right)^{2n}}{\left(\frac{1}{2}\right)^{2n}}$$

$$= \frac{2n-k}{k+1} \quad ✓$$

Solve $P_{k+1} > P_k$ for greatest value of $P_k$

ie $2n-k > k+1$

$2k < 2n-1$

$k < n - \frac{1}{2}$

$\therefore P_{k+1} > P_k$ ✓ reasoning.

for $k = 0, 1, 2, \ldots, n-1$

and $P_{k+1} < P_k$

for $k = n+1, n+2, \ldots 2n$

$\therefore k = n$ gives greatest value for $P_k$ and hence most likely.

ii) $P_n = {}^{2n}C_n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{2n-n}$

$$= \frac{2n!}{n!\,n!} \cdot \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^n$$

$$= \frac{2n!}{(n!)^2} \quad ✓$$

**2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS**

| Suggested Solution (s) | Comments |
|---|---|
| **Question 8** | |
| (a) i) Let $u=(1-\ln x)^n$ | $v'=1$ ✓1 |
| $\therefore u' = \dfrac{n(1-\ln x)^{n-1}}{-x}$ | $v=x$ |
| $\therefore I_n = \left[x(1-\ln x)^n\right]_1^e + n\displaystyle\int_1^e (1-\ln x)^{n-1}\,dx$ | ✓1 |
| $\therefore I_n = -1 + nI_{n-1}$ | |
| ii) $I_0 = \displaystyle\int_1^e 1\,dx = e-1$ | ✓ |
| $\therefore I_3 = -1+3I_2$ | |
| $= -1+3(-1+2I_1)$ | |
| $= -4+6(-1+I_0)$ | |
| $= -16+6e$ | ✓ |
| iii) $\dfrac{I_r}{r!} = \dfrac{-1}{r!} + \dfrac{rI_{r-1}}{r!}$ | |
| $= \dfrac{-1}{r!} + \dfrac{I_{r-1}}{(r-1)!}$ | ✓ |
| $\therefore \displaystyle\sum_{r=1}^{n}\frac{I_r}{r!} = -\sum_{r=1}^{n}\frac{1}{r!} + \sum_{r=1}^{n}\frac{I_{r-1}}{(r-1)!}$ | |
| $= -\displaystyle\sum_{r=1}^{n}\frac{1}{r!} + \sum_{r=0}^{n-1}\frac{I_r}{r!}$ | |
| $\therefore \dfrac{I_n}{n!} = -\displaystyle\sum_{r=1}^{n}\frac{1}{r!} + \frac{I_0}{0!}$ | |
| $= -\displaystyle\sum_{r=1}^{n}\frac{1}{r!} + \frac{-1}{0!} + \frac{e}{0!}$ | |
| $= e - \displaystyle\sum_{r=1}^{n}\frac{1}{r!}$ | ✓ reasoning to answer |
| iv) $1 \le x \le e$ | |
| $\therefore 0 \le \ln x \le 1$ | |
| $0 \le (1-\ln x)^n \le 1$ | |
| So $0 \le \displaystyle\int_1^e (1-\ln x)^n\,dx \le \int_1^e dx$ | |
| $0 \le$ [illegible] | |
| v) $0 \le \dfrac{I_n}{n!} \le \dfrac{e-1}{n!}$ | |
| as $n\to\infty$, $\dfrac{e-1}{n!} \to 0$ | |
| $\therefore \displaystyle\lim_{n\to\infty}\frac{I_n}{n!} = 0$ | ✓ and then deduce |

so $\displaystyle\lim_{n\to\infty}\left(e - \sum_{r=0}^{n}\frac{1}{r!}\right) = \lim_{n\to\infty}\frac{I_n}{n!} = 0$

$\therefore e - \displaystyle\lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{1}{r!}\right) = 0$

$\displaystyle\lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{1}{r!}\right) = e$

# 2009 Year 12 Mathematics Extension 2 Task 4 Trial HSC SOLUTIONS

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|

Q8 (b) i)

Since $a > 0$, $\int_0^a \{\lambda\phi(x) + \psi(x)\}^2 dx \geq 0$ for all real $\lambda$ ✓

$$\therefore \int_0^a \{\lambda\phi(x) + \psi(x)\}^2 dx = \lambda^2 \int_0^a \{\phi(x)\}^2 dx + 2\lambda \int_0^a \phi(x)\psi(x)\,dx + \int_0^a \{\psi(x)\}^2 dx$$ ✓

Considered as a quadratic in $\lambda$, $\Delta \leq 0$. ✓

$$\therefore 4\left\{\int_0^a \phi(x)\psi(x)\,dx\right\}^2 - 4\int_0^a \{\phi(x)\}^2 dx \cdot \int_0^a \{\psi(x)\}^2 dx \leq 0$$

$$\therefore \left\{\int_0^a \phi(x)\psi(x)\,dx\right\}^2 \leq \int_0^a \{\phi(x)\}^2 dx \cdot \int_0^a \{g(x)\}^2 dx$$ ✓

ii) if $a = 1$ and $\psi(x) = 1$ ✓

$\int_0^1 1^2 dx = 1$ ✓

$$\therefore \left\{\int_0^1 \phi(x)\,dx\right\}^2 \leq \int_0^1 \{\phi(x)\}^2 dx$$ ✓

iii) $$\left\{\int_0^1 \phi(x)\,dx\right\}^4 \leq \left\{\int_0^1 \{\phi(x)\}^2 dx\right\}^2 \leq \int_0^1 \left\{\{\phi(x)\}^2\right\}^2 dx$$

$$\therefore \left\{\int_0^1 f(x)\,dx\right\}^4 \leq \int_0^1 \{f(x)\}^4 dx.$$